

RENEWAL STRUCTURE OF THE BROWNIAN TAUT STRING

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ABSTRACT. In a recent paper [LS15], M. Lifshits and E. Setterqvist introduced the taut string of a Brownian motion w , defined as the function of minimal quadratic energy on $[0, T]$ staying in a tube of fixed width $h > 0$ around w . The authors showed a Law of Large Number (L.L.N.) for the quadratic energy spent by the string for a large time T .

In this note, we exhibit a natural renewal structure for the Brownian taut string, which is directly related to the time decomposition of the Brownian motion in terms of its h -extrema (as first introduced by Neveu and Pitman [NP89]). Using this renewal structure, we derive an expression for the constant in the L.L.N. given in [LS15]. In addition, we provide a Central Limit Theorem (C.L.T.) for the fluctuations of the energy spent by the Brownian taut string.

1. INTRODUCTION AND MAIN RESULTS

Let $AC([a, b])$ denote the set of absolutely continuous functions defined on $[a, b]$, and $\|f\|_{\infty, [a, b]}$ denote the supremum of the function $|f|$ over $[a, b]$. Given $T, h > 0$ and a continuous function w , the taut string associated with w is the function such that for every strictly convex function c , it is the unique solution of the following minimization problem

$$(1) \quad \text{Min} \left\{ \int_0^T c(\varphi'(u)) \, du : \varphi \in AC([0, T]), \varphi(0) = w(0), \varphi(T) = w(T), \|w - \varphi\|_{\infty, [0, T]} \leq \frac{h}{2} \right\},$$

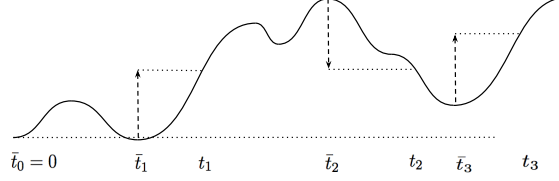
Interestingly, the solution of the latter minimization problem does not depend on the choice of c — see Proposition 6.2 for more details.

In a recent paper [LS15], M. Lifshits and E. Setterqvist studied the long time behavior of the taut string constructed around the sample path of a Brownian motion. Using an argument based on a concentration inequality for Gaussian processes, they showed that if w is a Wiener process sample path, then there exists a (non-explicit) constant \mathcal{C} such that

$$(2) \quad \lim_{T \uparrow \infty} \frac{1}{T} \int_0^T |\eta'_T(u)|^2 \, du = \mathcal{C} \quad \text{a.s.},$$

where η_T denotes the taut string associated with the path w on the interval $[0, T]$. As they put it, the constant \mathcal{C} “shows how much quadratic energy an absolutely continuous function must spend if it is bounded to stay within a certain distance from the trajectory of w ”. The aim of this note is to provide a generalization of their result. We will show that an analogous result holds for a large class of penalization function c , i.e. that for a very general class of functions c , there exists \mathcal{C}_c such that

$$(3) \quad \lim_{T \uparrow \infty} \frac{1}{T} \int_0^T c(\eta'_T(u)) \, du = \mathcal{C}_c \quad \text{a.s.}$$

FIGURE 1. Peaks and valleys of the function w .

In addition, we provide a semi-explicit expression for the constant \mathcal{C}_c , and an estimate for the fluctuations of the energy around this limiting value.

Our result is based on a decomposition of the Brownian motion in terms of its h -extrema that was first proposed by Neveu and Pitman [NP89] and that we now expose. Let us introduce two sequences of times $\{t_n(w)\}_{n \geq 0} \equiv \{t_n\}_{n \geq 0}$ and $\{\bar{t}_n(w)\}_{n \geq 0} \equiv \{\bar{t}_n\}_{n \geq 0}$: we first set $t_0, \bar{t}_0 = 0$, and for $n \geq 0$,

$$\begin{aligned} t_{2n+1} &= \inf\{t \geq t_{2n} : w(t) - \inf_{[t_{2n}, t]} w = h\} \\ t_{2n+2} &= \inf\{t \geq t_{2n+1} : \sup_{[t_{2n+1}, t]} w - w(t) = h\} \end{aligned}$$

and

$$\begin{aligned} \bar{t}_{2n+1} &= \sup\{t \in [t_{2n}, t_{2n+1}] : \inf_{[t_{2n}, t]} w = w(t)\} \\ \bar{t}_{2n+2} &= \sup\{t \in [t_{2n+1}, t_{2n+2}] : \sup_{[t_{2n+1}, t]} w = w(t)\} \end{aligned}$$

In other words, \bar{t}_{2n+1} (resp., \bar{t}_{2n+2}) is the starting time of the first upper (resp., lower) sub-excursion of height $\geq h$ after time t_{2n} (resp., t_{2n+1}), see Fig. 1. Following the terminology of Neveu and Pitman [NP89], the $w(\bar{t}_n)$'s correspond to the h -extrema of the function w , whereas the times t_n 's can be thought of as delimiting the successive peaks and valleys of height and depth greater than h formed by the path w . In [NP89], the authors proved that this decomposition is natural for the Brownian motion: for a Wiener sample path w , the sequence of paths $\{(-1)^n (w(\bar{t}_n + u) - w(\bar{t}_n)); 0 \leq u \leq \bar{t}_{n+1} - \bar{t}_n\}_{n \geq 1}$ is a sequence of i.i.d. random variables.

From Proposition 6.1 in the Appendix, for every $i \geq 1$, there exists a unique solution to the following minimization problem

$$\begin{aligned} &\text{Min}\left\{\int_{\bar{t}_i}^{\bar{t}_{i+1}} |\varphi'(u)|^2 du : \varphi \in \text{AC}(I_i), \right. \\ &\left. \varphi(\bar{t}_i) = w(\bar{t}_i) - (-1)^i \frac{h}{2}, \varphi(\bar{t}_{i+1}) = w(\bar{t}_{i+1}) + (-1)^i \frac{h}{2}, \|\varphi - w\|_{\infty, I_i} \leq \frac{h}{2}\right\}, \end{aligned}$$

where $I_i := [\bar{t}_i, \bar{t}_{i+1}]$. We denote this element by ψ_i . The following theorem shows that the Neveu and Pitman's decomposition is also natural for the taut string.

Theorem 1.1. *Let w be a continuous function (deterministic or random) and let η_T be the associated taut string on $[0, T]$. Let*

$$N(T) = \sup\{n : \bar{t}_n \leq T\}.$$

If $N(T) \geq 4$, the taut string η_T and the function ψ_i coincide on $[\bar{t}_i, \bar{t}_{i+1}]$ for every $2 \leq i \leq N(T) - 2$.

We note in passing that the previous result implies the \bar{t}_i 's for $2 \leq i \leq N(T) - 1$ are knot points for the taut string, i.e., that the string hits the boundary of the tube at those times.

Let w be the sample path of a Brownian motion. Since the sequence

$$\{(-1)^n (w(\bar{t}_n + u) - w(\bar{t}_n)); 0 \leq u \leq \bar{t}_{n+1} - \bar{t}_n\}_{n \geq 1}$$

is a sequence of i.i.d random variables, the latter result implies (at least informally) that the Brownian taut string on $[\bar{t}_2, \bar{t}_{N(T)} - 1]$ is obtained by pasting together independent copies of the random path ψ_1 (up to a change of sign). Using standard limit theorems, we can then leverage this intrinsic renewal structure to derive a L.L.N. and a C.L.T. for the long time behavior of the energy spent by Brownian taut string.

Theorem 1.2. *Let $h, T > 0$. Let c be a locally bounded function, and $\alpha \geq 0$ be such that $\lim_{|x| \rightarrow \infty} c(x)/x^\alpha = 0$. Define*

$$\begin{aligned} \tau &= \bar{t}_4 - \bar{t}_2, \\ \mathcal{E} &= \int_{\bar{t}_2}^{\bar{t}_3} c(\psi_2'(u)) du + \int_{\bar{t}_3}^{\bar{t}_4} c(\psi_3'(u)) du. \end{aligned}$$

If w be the sample path of a Brownian motion, then

- (1) τ and \mathcal{E} have all finite moments.
- (2) $\frac{1}{T} \int_0^T c(\eta_T'(u)) du \rightarrow E(\mathcal{E})/E(\tau)$ a.s..
- (3)

$$\frac{1}{\sqrt{T} \hat{\sigma}^2} \left(\int_0^T c(\eta_T'(u)) du - T \frac{E(\mathcal{E})}{E(\tau)} \right) \rightarrow \mathcal{Z} \text{ in distribution,}$$

where \mathcal{Z} is a standard normal random variable and

$$\hat{\sigma}^2 = \frac{(E(\mathcal{E}))^2}{(E(\tau))^3} \text{Var}(\tau) + \frac{\text{Var}(\mathcal{E})}{E(\tau)} - 2 \text{Cov}(\tau, \mathcal{E}) \frac{E(\mathcal{E})}{(E(\tau))^2}$$

Outline of the paper. The rest of the paper will be organized as follows. In Section 2, we give a proof of Theorem 1.1 and show some technical results that will be useful to control the moments of \mathcal{E} and τ . In Section 1.2, we give an outline of the proof of Theorem 1.2, and postpone technical details to later sections. More precisely, in Section 4, we control the moments of \mathcal{E} and τ and show some tightness results. In Section 5, we prove an extension of the C.L.T. for renewal processes. Finally, in the Appendix, we show some results related to the taut string in a deterministic setting.

2. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. Along the way, we also prove a result that will be instrumental in controlling the moments of τ and \mathcal{E} . In the following, \bar{c} will denote an even, strictly convex function.

Lemma 2.1. *Let f be a continuous function such that $\bar{t}_1(f) = 0$. Let $V \geq t_1(f)$ and let Ψ be the unique minimizer of*

$$\text{Min}\left\{\int_0^V \bar{c}(\varphi'(u)) du : \varphi \in AC([0, V]), \|f - \varphi\|_{\infty, [0, V]} \leq \frac{h}{2}\right\}.$$

Then $\Psi(0) = f(0) + \frac{h}{2}$

Proof. The existence and uniqueness of Ψ are given by Proposition 6.1 in the Appendix. Next, we need to show that for any absolutely continuous function φ , such that $\|\varphi - f\|_{\infty, [0, V]} \leq h/2$, we can construct an absolutely continuous function $\bar{\varphi}$ such that (1) $\bar{\varphi}$ stays in the tube of width h around f (i.e., such that $\|\bar{\varphi} - f\|_{\infty, [0, V]} \leq h/2$), (2) we have the following boundary condition

$$\bar{\varphi}(0) = f(0) + h/2$$

and finally, (3) $\bar{\varphi}$ has lower \bar{c} -energy: $\int_0^V \bar{c}(\bar{\varphi}'(u)) du \leq \int_0^V \bar{c}(\varphi'(u)) du$.

We first claim that for every admissible function φ (i.e., φ is AC and satisfies condition (1) above), there exists $s \in [0, t_1(f)]$ such that $\varphi(s) = f(0) + h/2$. This simply follows from the fact that for every $\varphi \in AC([0, V])$

$$\varphi(0) \leq f(0) + h/2 \text{ and } \varphi(t_1(f)) \geq f(t_1(f)) - h/2 = f(0) + h/2,$$

where we used the fact that $\bar{t}_1(f) = 0$. This yields the existence of s in $[0, t_1(f)]$ such that $\varphi(s) = f(0) + h/2$, as claimed earlier.

Let us now consider the absolutely continuous $\bar{\varphi}$ defined on $[0, V]$ as follows:

$$\bar{\varphi}(u) = 1_{u \in [0, s]}(f(0) + h/2) + 1_{u \in [s, V]} \varphi(u).$$

From the definition of $t_1(f)$, it is straightforward to check that this function is guaranteed to stay in the tube of width h . Furthermore, since the minimum of \bar{c} is attained at 0, we have:

$$\int_0^V \bar{c}(\bar{\varphi}'(u)) du \leq \int_0^V \bar{c}(\varphi'(u)) du,$$

which ends the proof of our lemma. □

Corollary 2.2. *For every $i \geq 1$, ψ_i is the unique minimizer of the free-boundary problem*

$$\text{Min}\left\{\int_{\bar{t}_i}^{\bar{t}_{i+1}} \bar{c}(\varphi'(u)) du : \varphi \in AC(I_i), \|w - \varphi\|_{\infty, I_i} \leq \frac{h}{2}\right\},$$

Proof. Let Q be the minimizer of the minimization problem described above (whose existence and uniqueness is again guaranteed by Proposition 6.1.). We claim that it is enough to show that

$$(4) \quad Q(\bar{t}_i) = w(\bar{t}_i) - (-1)^i \frac{h}{2}, \text{ and } Q(\bar{t}_{i+1}) = w(\bar{t}_{i+1}) + (-1)^i \frac{h}{2}.$$

In order to see this, we first note that those identities will directly imply that Q coincides with the unique element in

$$\begin{aligned} & \text{Argmin}\left\{\int_{\bar{t}_i}^{\bar{t}_{i+1}} \bar{c}(\varphi'(u)) du : \varphi \in \text{AC}(I_i), \right. \\ & \left. \varphi(\bar{t}_i) = w(\bar{t}_i) - (-1)^i \frac{h}{2}, \varphi(\bar{t}_{i+1}) = w(\bar{t}_{i+1}) + (-1)^i \frac{h}{2}, \|\varphi - w\|_{\infty, I_i} \leq \frac{h}{2}\right\}. \end{aligned}$$

By Proposition 6.2 in the Appendix, this implies that Q is also the unique minimizer of the minimization problem obtained by replacing \bar{c} by any strictly convex function c in the latter expression. In particular, it is solution of the following minimization problem:

$$\begin{aligned} & \text{Min}\left\{\int_{\bar{t}_i}^{\bar{t}_{i+1}} |\varphi'(u)|^2 du : \varphi \in \text{AC}(I_i), \right. \\ & \left. \varphi(\bar{t}_i) = w(\bar{t}_i) - (-1)^i \frac{h}{2}, \varphi(\bar{t}_{i+1}) = w(\bar{t}_{i+1}) + (-1)^i \frac{h}{2}, \|\varphi - w\|_{\infty, I_i} \leq \frac{h}{2}\right\}, \end{aligned}$$

which yields $\psi_i = Q$.

It remains to show the two identities in (4). We start by showing the first equality. Define $f(u) = (-1)^{i+1} w(u + \bar{t}_i)$ for $u \in [0, \bar{t}_{i+1} - \bar{t}_i]$. In the following, we will use the following notation

$$\theta_i = t_i(f), \quad \bar{\theta}_i = \bar{t}_i(f)$$

(whereas we recall that the \bar{t}_i 's are defined with respect to the function w , i.e., $\bar{t}_i \equiv \bar{t}_i(w)$). Note that under those notations, we have $\theta_1 = 0$ and $\bar{\theta}_2 = \bar{t}_{i+1} - \bar{t}_i$. Let us now consider Ψ the unique minimizer of

$$\text{Min}\left\{\int_0^{\bar{\theta}_2} \bar{c}(\varphi'(u)) du : \varphi \in \text{AC}([0, \bar{\theta}_2]), \|\varphi - f\|_{\infty, [0, \bar{\theta}_2]} \leq \frac{h}{2}\right\}.$$

and let $\tilde{\Psi}(u) = (-1)^{i+1} \Psi(u - \bar{t}_i)$ for every $u \in [\bar{t}_i, \bar{t}_{i+1}]$. Using the fact that \bar{c} is even, one can readily check from the definition that $\tilde{\Psi}$ belongs to

$$\text{Argmin}\left\{\int_{\bar{t}_i}^{\bar{t}_{i+1}} \bar{c}(\varphi'(u)) du : \varphi \in \text{AC}([\bar{t}_i, \bar{t}_{i+1}]), \|\varphi - w\|_{\infty, [\bar{t}_i, \bar{t}_{i+1}]} \leq \frac{h}{2}\right\}$$

and is therefore the unique minimizer of the corresponding variational problem. As a consequence, it coincides with Q . On the other hand, since $\bar{\theta}_2 > \theta_1$ and $\bar{\theta}_1 = 0$, the previous lemma implies $\Psi(0) = f(0) + \frac{h}{2}$ and thus

$$Q(\bar{t}_i) = (-1)^{i+1} \Psi(0) = w(\bar{t}_i) + (-1)^{i+1} \frac{h}{2}.$$

Let us now turn to the second equality. Again, the idea is to use the previous lemma on a suitably chosen function. Define $g(u) = (-1)^i w(\bar{t}_{i+1} - u)$ on $[0, \bar{t}_{i+1} - \bar{t}_i]$. Finally, define $\tau_i = t_i(g)$ and $\bar{\tau}_i = \bar{t}_i(g)$. By construction, $\bar{\tau}_1 = 0$ and it is straightforward to check that $\bar{t}_{i+1} - \bar{t}_i \geq \tau_1$. Thus, denoting by $\hat{\Psi}$ the unique solution of the minimization problem

$$\text{Min}\left\{\int_0^{\bar{\tau}_{i+1} - \bar{\tau}_i} \bar{c}(\varphi'(u)) du : \varphi \in \text{AC}([0, \bar{t}_{i+1} - \bar{t}_i]), \|\varphi - g\|_{\infty, [0, \bar{t}_{i+1} - \bar{t}_i]} \leq \frac{h}{2}\right\},$$

we must have $\hat{\Psi}(0) = g(0) + \frac{h}{2}$. On the other hand, the function $(-1)^i \hat{\Psi}(\bar{t}_{i+1} - u)$ on the interval $[\bar{t}_i, \bar{t}_{i+1}]$ coincides with Q (by the same argument as above). Combining this fact with $\hat{\Psi}(0) = g(0) + \frac{h}{2}$ yields the desired result. \square

Proof of Theorem 1.1. Let us consider Φ defined on $[\bar{t}_1, \infty)$ that coincides with ψ_i on the interval $[\bar{t}_i, \bar{t}_{i+1})$ for $i \geq 1$. We claim that the function Φ is a solution of the following free-boundary minimization problem:

$$\text{Min}\left\{\int_{\bar{t}_1}^{\bar{t}_{N(T)}} |\varphi'(u)|^2 du : \varphi \in \text{AC}([0, T]), \|w - \varphi\|_{\infty, [0, T]} \leq \frac{h}{2}\right\}.$$

for every T such that $N(T) \geq 1$. First, the condition $\|w - \Phi\|_{\infty, [0, T]} \leq h/2$ is obviously satisfied. Further, Φ is absolutely continuous on $[\bar{t}_1, \infty]$ because of the boundary conditions imposed on the ψ 's. Furthermore, for every $\varphi \in \text{AC}([\bar{t}_1, \bar{t}_{N(T)}])$:

$$\begin{aligned} \int_{\bar{t}_1}^{\bar{t}_{N(T)}} |\varphi'(u)|^2 du &= \sum_{i=1}^{N(T)-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} |\varphi'(u)|^2 du \\ &\geq \sum_{i=1}^{N(T)-1} \int_{\bar{t}_i}^{\bar{t}_{i+1}} |\psi'_i(u)|^2 du \\ &= \int_{\bar{t}_1}^{\bar{t}_{N(T)}} |\Phi'(u)|^2 du \end{aligned}$$

where the inequality follows from Corollary 2.2 with $\bar{c}(x) = x^2$. Hence, Φ is a solution of the free-boundary minimization problem described above.

Next, we claim that for every interval $[\bar{t}_i, \bar{t}_{i+1}]$, $1 \leq i \leq N(T) - 1$, there must exist at least one $v_i \in [\bar{t}_i, \bar{t}_{i+1}]$ such that $\eta_T(v_i) = \Phi(v_i)$. Let us first assume that i is even. Since at \bar{t}_i (resp., \bar{t}_{i+1}), the function ψ_i is at the lower (resp., upper) boundary of the tube of width h surrounding w , we must have

$$\eta_T(\bar{t}_i) - \Phi(\bar{t}_i) \geq 0, \quad \text{and} \quad \eta_T(\bar{t}_{i+1}) - \Phi(\bar{t}_i) \leq 0.$$

and the previous claim flows in the case where i is even. The odd case can be treated along the same lines.

Next, recall that the taut string η_T is the unique solution of the minimization problem:

$$\text{Min}\left\{\int_0^T |\varphi'(u)|^2 du : \varphi \in \text{AC}([0, T]), \varphi(0) = w(0), \varphi(T) = w(T), \|w - \varphi\|_{\infty, [0, T]} \leq \frac{h}{2}\right\}.$$

Thus, the restriction of η_T and Φ on the interval $[v_1, v_{N(T)-1}]$ must be solution of the following minimization problem :

$$\begin{aligned} &\text{Min}\left\{\int_{v_1}^{v_{N(T)-1}} |\varphi'(u)|^2 du : \varphi \in \text{AC}([v_1, v_{N(T)-1}]), \right. \\ &\varphi(v_1) = \eta_T(v_1) = \Phi(v_1), \quad \varphi(v_{N(T)-1}) = \eta_T(v_{N(T)-1}) = \Phi(v_{N(T)-1}), \\ &\left. \|w - \varphi\|_{\infty, [v_1, v_{N(T)-1}]} \leq \frac{h}{2}\right\}. \end{aligned}$$

Since there is a unique solution to the latter minimization problem (again by Proposition 6.1), it follows that η_T and Φ coincide on the interval $[v_1, v_{N(T)-1}]$. Finally, since $[\bar{t}_2, \bar{t}_{N(T)-1}] \subset [v_1, v_{N(T)-1}]$, this ends the proof of Theorem 1.2. \square

3. PROOF OF THEOREM 1.2

In this section, we give an outline of the proof of Theorem 1.2, and postpone technical details to later sections. First, it is sufficient to show a weak version of our L.L.N.. Indeed, one can extend our convergence in probability statement to convergence a.s. statement by using the same argument presented in Theorem 1.2. in [LS15].

When $N(T) \geq 4$, Theorem 1.1 implies that

$$\int_0^T c(\eta'_T(u))du = \int_0^{\bar{t}_2} c(\eta'_T(u))du + \sum_{i=2}^{N(T)-2} \int_{\bar{t}_i}^{\bar{t}_{i+1}} c(\psi'_i(u))du + \int_{\bar{t}_{N(T)-1}}^T c(\eta'_T(u))du.$$

Let us define

$$Y_i = \int_{\bar{t}_{2i}}^{\bar{t}_{2i+1}} c(\psi'_{2i}(u))du + \int_{\bar{t}_{2i+1}}^{\bar{t}_{2i+2}} c(\psi'_{2i+1}(u))du$$

and let $\mathcal{N}(T) = \sup\{n : \bar{t}_{2n} \leq T\}$. Again assuming that $N(T) \geq 4$, a little bit of algebra yields

$$\int_0^T c(\eta'_T(u))du = \sum_{i=1}^{\mathcal{N}(T)-1} Y_i + R(T),$$

with

$$\begin{aligned} R(T) &= \int_0^{\bar{t}_2} c(\eta'_T(u))du + \int_{\bar{t}_{N(T)-1}}^T c(\eta'_T(u))du \\ (5) \quad &- 1_{\{N(T) \text{ is even}\}} \int_{\bar{t}_{N(T)-1}}^{\bar{t}_{N(T)}} c(\psi'_{N(T)-1}(u))du. \end{aligned}$$

In Section 4, we show that $\{R(T)\}_{T \geq 0}$ is tight – see Corollary 4.5 – which implies that $R(T)/\sqrt{T}$ and $R(T)/T$ converges to 0 in probability as $T \rightarrow \infty$. As a consequence, we only need to show the L.L.N. and the C.L.T. for the quantity $\sum_{i=1}^{\mathcal{N}(T)-1} Y_i$. We start with the L.L.N. (second item of Theorem 1.2). Write

$$\frac{1}{T} \sum_{i=1}^{\mathcal{N}(T)-1} Y_i = \left(\frac{1}{\mathcal{N}(T)} \sum_{i=1}^{\mathcal{N}(T)-1} Y_i \right) \frac{\mathcal{N}(T)}{T}.$$

First, $\{\bar{t}_{2i+2} - \bar{t}_{2i}\}_{i \geq 1}$ is a sequence of i.i.d. random variables, and in Corollary 4.3, we shall prove that its elements have all finite moments. A standard renewal theorem implies that $\frac{\mathcal{N}(T)}{T}$ converges a.s. to $1/E(\bar{t}_4 - \bar{t}_2)$.

Secondly, the sequence $\{(-1)^i(w(u + \bar{t}_i) - w(\bar{t}_i)), u \in [0, \bar{t}_{i+1} - \bar{t}_i]\}_{i \geq 1}$ is a sequence of i.i.d. random variables. This implies that $\{Y_i\}_{i \geq 1}$ is also made of i.i.d. random variables. Finally, we shall prove later that Y_i has all finite moments – see again Corollary 4.3 below. Our L.L.N. then follows by a direct application of the strong L.L.N. and by noting that $\mathcal{N}(T)$ goes to ∞ as $T \rightarrow \infty$.

In order to prove our C.L.T., we will need to prove a result on renewal processes that we now expose. Let $\{(X_i, \tau_i)\}$ be an i.i.d. sequence of (possibly correlated) pairs of non-negative random variables with respective finite non-zero expected value $\bar{\tau}$ and \bar{X} , finite and non-zero standard deviation σ_τ and σ_X and covariance $\sigma_{X,\tau}$. Define

$$S_n = \sum_{k=1}^n \tau_k, \text{ and } U_n = \sum_{k=1}^n X_k.$$

and $\tilde{\mathcal{N}}(t) = \sup\{n \geq 0 : S_n \leq t\}$. The following result is an extension of Anscombe Theorem [A52].

Proposition 3.1.

$$\lim_{t \rightarrow \infty} \frac{\left(U_{\tilde{\mathcal{N}}(t)-1} - t\bar{X}/\bar{\tau} \right)}{\sqrt{t\bar{\sigma}^2}} = \mathcal{Z} \text{ in distribution,}$$

where \mathcal{Z} is a standard normal random variable and

$$\bar{\sigma}^2 = \frac{\bar{X}^2}{\bar{\tau}^3} \sigma_\tau^2 + \frac{\sigma_X^2}{\bar{\tau}} - 2\sigma_{X,\tau} \frac{\bar{X}}{\bar{\tau}^2}$$

Taking $X_i = \int_{\bar{t}_{2i}}^{\bar{t}_{2i+1}} c(\psi'_{2i}(u))du + \int_{\bar{t}_{2i+1}}^{\bar{t}_{2i+2}} c(\psi'_{2i+1}(u))du$ and $\tau_i = \bar{t}_{2i+2} - \bar{t}_{2i}$ in the latter proposition, yields the third part of Theorem 1.2.

4. MOMENT ESTIMATES AND TIGHTNESS.

4.1. Moments of \mathcal{E} and τ . We start with some preliminary work. Define $\sigma_0 = 0$ and for $n \geq 1$,

$$\sigma_n = \inf\{t \geq \sigma_{n-1} : |w(t) - w(\sigma_{n-1})| = h/4\}.$$

By the strong Markov property, we note that $(\sigma_n - \sigma_{n-1}; n \geq 1)$ is a sequence of i.i.d. random variables.

Lemma 4.1. *For every $p \in \mathbb{Z}$, $E((\sigma_1)^p) < \infty$.*

Proof. The case $p \in \mathbb{N}$ is well known. Let us focus instead on the case where $p < 0$, i.e., for $n \in \mathbb{N}$, let us prove that $E((1/\sigma_1)^n) < \infty$. First,

$$\begin{aligned} E(1/(\sigma_1)^n) &= n \int_0^\infty x^{n-1} P(1/\sigma_1 > x) dx \\ &= n \int_0^\infty x^{n-1} P(\sigma_1 < \frac{1}{x}) dx. \end{aligned}$$

We then need to estimate the asymptotic behavior of $P(\sigma_1 < y)$ when y goes to 0. Momentarily, we make the dependence in h explicit in the notations by writing $\sigma_n = \sigma_n^h$. By Brownian scaling and symmetry,

$$\begin{aligned} P(\sigma_1^h < y) &= P(\sigma_1^{h/\sqrt{y}} < 1) \\ &\leq 2P(\max_{[0,1]} w \geq \frac{h}{4\sqrt{y}}). \end{aligned}$$

By standard large deviation estimates,

$$\lim_{y \rightarrow 0} 16y/h^2 \log P(\max_{[0,1]} w \geq \frac{h}{4\sqrt{y}}) = -\frac{1}{2}.$$

It follows that $P(\sigma_1^h < y)$ decreases exponentially fast to 0 as $y \rightarrow 0$, and thus that $E(1/(\sigma_1^h)^n)$ is finite. \square

For $n \geq 1$, define

$$\Delta_n = \frac{4}{h}(w(\sigma_n) - w(\sigma_{n-1})).$$

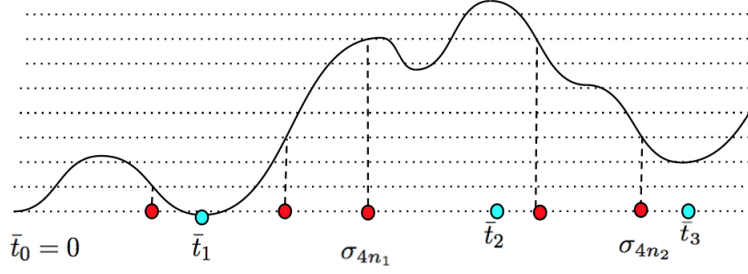


FIGURE 2. Red points represent time points of the form σ_{4k} . In this example, $n_1 = 3$ and $n_2 = 5$.

$\{\Delta_n, n \geq 1\}$ is a sequence of i.i.d. random variables, independent of the σ_n 's, and $P(\Delta_n = \pm 1) = 1/2$. From the sequence $\{\Delta_n\}_{n \geq 1}$, define a sequence of integer $\{n_k\}_{k \geq 0}$ as follows: $n_0 = 0$ and

$$n_k = \inf\{i \geq n_{k-1} : \Delta_{4i} = \Delta_{4i-1} = \Delta_{4i-2} = \Delta_{4i-3} = (-1)^{k+1}\}.$$

Note that if $\Delta_{4n_k} = 1$ (resp., $\Delta_{4n_k} = -1$), the path w experiences an upcrossing (resp., downcrossing) of size h on the interval $[\sigma_{4(n_k-1)}, \sigma_{4n_k}]$. In particular, on the interval $[0, \sigma_{4n_2}]$, the path w must have experienced an upcrossing and then a downcrossing of size larger or equal to h . See Fig. 2. This motivates the following lemma.

Lemma 4.2.

$$\bar{t}_2 - \bar{t}_1 \leq \sigma_{4n_2}.$$

Proof. Let i_k be the index such that $\sigma_{4n_k} \in [\bar{t}_{i_k}, \bar{t}_{i_k+1})$. It is sufficient to show that (1) $i_1 \geq 1$, and (2) $i_1 \neq i_2$.

Let us first deal with (1). By definition of n_1 , we must have $w(\sigma_{4n_1}) - \inf_{[0, \sigma_{4n_1}]} w \geq h$, which implies that $t_1 \leq \sigma_{4n_1}$. Since $\bar{t}_1 \leq t_1$, we get that $\bar{t}_1 \leq \sigma_{4n_1}$ and thus $i_1 \geq 1$.

We now proceed with (2). We claim that if k is such that $\Delta_{4n_k} = -1$, then σ_{4n_k} does not belong to $[\bar{t}_i, \bar{t}_{i+1})$ for every odd integer i . Recall that if i is odd

$$t_{i+1} = \inf\{u \geq t_i : \sup_{[t_i, u]} w - w(u) = h\}, \text{ and } \bar{t}_{i+1} = \sup\{u \geq t_i : \sup_{[t_i, u]} w = w(u)\}$$

On the one hand, it is easy to check that w attains its only minimum at \bar{t}_i on the interval $[\bar{t}_i, \bar{t}_{i+1}]$. On the other hand, on the interval $[\sigma_{4(n_k-1)}, \sigma_{4n_k}]$, w attains its minimum at σ_{4n_k} since $\Delta_{\sigma_{4n_k}} = \dots = \Delta_{\sigma_{4n_k-3}} = -1$. Thus, if $\sigma_{4n_k} \in [\bar{t}_i, \bar{t}_{i+1}]$, we would have $\sigma_{4(n_k-1)} \geq \bar{t}_i$ and

$$\begin{aligned} \sup_{[t_i, \sigma_{4n_k}]} w - w(\sigma_{4n_k}) &= \sup_{[\bar{t}_i, \sigma_{4n_k}]} w - w(\sigma_{4n_k}) \\ &\geq \sup_{[\sigma_{4(n_k-1)}, \sigma_{4n_k}]} w - w(\sigma_{4n_k}) = h, \end{aligned}$$

where the first equality follows from the fact that $\sup_{[\bar{t}_i, t_i]} w = w(t_i)$ since i is odd. The latter inequality would imply that $\sigma_{4n_k} \geq t_{i+1} > \bar{t}_{i+1}$, thus yielding a contradiction.

By a symmetric argument, if k is such that $\Delta_{4n_k} = 1$, then $\sigma_{4n_k} \notin [\bar{t}_i, \bar{t}_{i+1}]$ when i even. By definition of n_1 and n_2 , $\Delta_{\sigma_{4n_1}} \neq \Delta_{\sigma_{4n_2}}$ and thus, σ_{4n_1} and σ_{4n_2} can not belong to the same interval $[\bar{t}_i, \bar{t}_{i+1}]$. This achieves the proof of claim (2) made earlier, and the proof of our lemma. \square

Corollary 4.3. *The random variables*

$$\tau = \bar{t}_4 - \bar{t}_2 \quad \text{and} \quad \mathcal{E} = \int_{\bar{t}_2}^{\bar{t}_3} c(\psi'_2(u))du + \int_{\bar{t}_3}^{\bar{t}_4} c(\psi'_3(u))du$$

have all finite moments.

Proof. First, $\bar{t}_4 - \bar{t}_2$ is equal in distribution to the sum of two independent copies of $\bar{t}_2 - \bar{t}_1$. Thus, controlling the moments of $\bar{t}_4 - \bar{t}_2$ amounts to controlling the moments of $\bar{t}_2 - \bar{t}_1$. Let $p \in \mathbb{N}$. From the previous lemma, we have

$$E(\bar{t}_2 - \bar{t}_1)^p \leq E(\sigma_{4n_2}^p).$$

By the strong Markov property and by symmetry, the random variable σ_{4n_2} is equal in distribution to the sum of 2 independent copies of σ_{4n_1} . Thus, it is enough to show that $E((\sigma_{4n_1})^p) < \infty$. Next,

$$(6) \quad E((\sigma_{4n_1})^p) = \sum_{n \geq 0} E[(\sum_{i=1}^{4n} \sigma_i)^p] P(n_1 = n),$$

where we used the independence between the n_k 's and the σ_i 's. It is straightforward to check that $\{n_k - n_{k-1}\}_{k \geq 1}$ is a sequence of independent geometric random variables with parameter $(1/2)^4$. It then follows that the tail of n_1 decreases exponentially fast. Since σ_1 has all finite moments (by Lemma 4.1), the latter identity implies that $\bar{t}_2 - \bar{t}_1$ also has all finite moments.

Let us now proceed with the second term. We will show that $\int_{\bar{t}_1}^{\bar{t}_2} c(\psi'_1(u))du$ has all finite moments. The moments of $\int_{\bar{t}_2}^{\bar{t}_3} c(\psi'_2(u))du$ can be controlled along the same lines. Recall that we made the assumption that c is locally bounded and that there exists $\alpha > 0$ such that

$$\lim_{|x| \rightarrow \infty} c(x)/|x|^\alpha = 0.$$

We will assume without loss of generality that $\alpha > 1$. Let $m, M \geq 0$ be such that $|c(x)| \leq m + M|x|^\alpha$ for every $x \in \mathbb{R}$, so that

$$|\int_{\bar{t}_1}^{\bar{t}_2} c(\psi'_1(u))du| \leq m(\bar{t}_2 - \bar{t}_1) + M \int_{\bar{t}_1}^{\bar{t}_2} |\psi'_1(u)|^\alpha du.$$

Since $\bar{t}_2 - \bar{t}_1$ has all finite moments, it remains to control the moments of $\int_{\bar{t}_1}^{\bar{t}_2} |\psi'_1(u)|^\alpha du$. In order to deal with this term, we will use the so-called free-knot approximation introduced in [LS15]: Let us consider the function ϕ obtained by linear interpolation of the points $(\sigma_n, w(\sigma_n))_{n \geq 0}$. In particular, this function is constructed in such a way that $\|\phi - w\|_{\infty, [0, \infty)} \leq h/2$. From Corollary 2.2, ψ_1 is the unique minimizer of

$$\text{Argmin}\left\{ \int_{\bar{t}_1}^{\bar{t}_2} |\varphi'(u)|^\alpha du : \varphi \in \text{AC}([\bar{t}_1, \bar{t}_2]), \|w - \varphi\|_{\infty, I_1} \leq \frac{h}{2} \right\},$$

and thus

$$\begin{aligned} \int_{\bar{t}_1}^{\bar{t}_2} |\psi'_1(u)|^\alpha du &\leq \int_{\bar{t}_1}^{\bar{t}_2} |\phi'(u)|^\alpha du \\ &\leq \int_0^{\sigma_{4n_2}} |\phi'(u)|^\alpha du, \end{aligned}$$

where the last inequality is a direct consequence of Lemma 4.2. Further,

$$\int_0^{\sigma_{4n_2}} |\phi'(u)|^\alpha du = \left(\frac{h}{4}\right)^\alpha \sum_{i=1}^{4n_2} \frac{1}{(\sigma_i - \sigma_{i-1})^{\alpha-1}}.$$

This yields for any $p \in \mathbb{N}$

$$E \left(\left(\int_{\bar{t}_1}^{\bar{t}_2} |\psi'_1(u)|^\alpha du \right)^p \right) \leq \left(\frac{h}{4}\right)^{\alpha p} E \left(\left(\sum_{i=1}^{4n_2} \frac{1}{(\sigma_i - \sigma_{i-1})^{\alpha-1}} \right)^p \right).$$

Using Lemma 4.1 and again the fact that $n_1, n_2 - n_1$ are independent geometric random variables independent of the σ_i 's (as in (6)), we get that $\int_{\bar{t}_1}^{\bar{t}_2} |\psi'_1(u)|^\alpha du$ has all finite moments. This ends the proof of the lemma. \square

We now turn to the tightness of $\{R(T)\}_{T \geq 0}$ as defined in (5).

4.2. Tightness of $R(T)$.

Lemma 4.4. *The sequences of random variables*

$$\left\{ \int_{\bar{t}_{N(T)-1}}^T c(\eta'_T(u)) du \right\}_{T \geq 0} \quad \text{and} \quad \left\{ \int_{\bar{t}_{N(T)-1}}^{\bar{t}_{N(T)}} c(\psi'_{N(T)-1}(u)) du \right\}_{T \geq 0}$$

are tight.

Proof. Step 0. We start by recalling a standard result from renewal theory. Define $\Delta \bar{t}_i = \bar{t}_{i+1} - \bar{t}_i$. $\Delta \bar{t}_i$ is obviously non-lattice and from the previous subsection, it has a finite first finite moment. Since the $\Delta \bar{t}_i$'s form a sequence of i.i.d. random variables, from [M74], the random sequence

$$(T - \bar{t}_{N(T)}, \Delta \bar{t}_{N(T)}, \dots, \Delta \bar{t}_{(N(T)-k)+}, \dots)$$

converges (in the sense of finite dimensional distributions) to the sequence

$$(u, \delta \bar{t}_0, \dots, \delta \bar{t}_k, \dots)$$

where the $\delta \bar{t}_i$'s are independent; for $i \geq 1$ the r.v. $\delta \bar{t}_i$ is distributed as $\Delta \bar{t}_2$; $\delta \bar{t}_0$ has the size biased distribution:

$$E(f(\delta \bar{t}_0)) = \frac{1}{E(\Delta \bar{t}_2)} \int_{\mathbb{R}^+} f(u) P(\Delta \bar{t}_2 > u) du$$

for every test function f (i.e., infinitely differentiable with compact support). Finally, u is independent of the $\delta \bar{t}_i$'s with $i \geq 1$, but conditioned on $\delta \bar{t}_0$, u is a uniform random variable on $[0, \delta \bar{t}_0]$ (in the renewal terminology, u has the backward recurrence time distribution).

Step 1. There exists $m, M \geq 0$ and $\alpha > 1$ such that $c(u) \leq m + M|u|^\alpha$. Thus

$$\left| \int_{\bar{t}_{N(T)-1}}^T c(\eta'_T(u)) du \right| \leq m(T - \bar{t}_{N(T)-1}) + M \int_{\bar{t}_{N(T)-1}}^T |\eta'_T(u)|^\alpha du.$$

Further,

$$\begin{aligned}
\left| \int_{\bar{t}_{N(T)-1}}^{N(T)} c(\psi'_{N(T)-1}(u)) du \right| &\leq m(\bar{t}_{N(T)} - \bar{t}_{N(T)-1}) + M \int_{\bar{t}_{N(T)-1}}^{\bar{t}_{N(T)}} |\psi'_{N(T)-1}(u)|^\alpha du \\
&\leq m(\bar{t}_{N(T)} - \bar{t}_{N(T)-1}) + M \int_{\bar{t}_{N(T)-1}}^{\bar{t}_{N(T)}} |\eta'_T(u)|^\alpha du \\
&\leq m(T - \bar{t}_{N(T)-1}) + M \int_{\bar{t}_{N(T)-1}}^T |\eta'_T(u)|^\alpha du.
\end{aligned}$$

where the second inequality is a direct consequence of Corollary 2.2. Thus, the two sequences of interest are bounded from above by the RHS of the latter inequality. Finally, since Step 0 above implies that the first term converges in distribution to $m(u + \delta t_1)$, it remains to show the tightness of $\{\int_{\bar{t}_{N(T)-1}}^T |\eta'_T(u)|^\alpha du\}_{T \geq 0}$.

Step 2. Define $\theta(T) = \sup\{n : \sigma_n \leq T\}$ and consider the function f_T obtained by linear interpolation of the points

$$\{(\sigma_k, w(\sigma_k))\}_{k \leq \theta(T)} \text{ and } (T, w(T)),$$

in such a way that f_T is an admissible function for the minimization problem (1). From Theorem 1.1, for $N(T) \geq 4$, we have

$$\begin{aligned}
\eta_T(\bar{t}_{N(T)-2}) - f_T(\bar{t}_{N(T)-2}) &= w(\bar{t}_{N(T)-2}) - f_T(\bar{t}_{N(T)-2}) - (-1)^{N(T)-2} \frac{h}{2} \\
\eta_T(\bar{t}_{N(T)-1}) - f_T(\bar{t}_{N(T)-1}) &= w(\bar{t}_{N(T)-1}) - f_T(\bar{t}_{N(T)-1}) - (-1)^{N(T)-1} \frac{h}{2}
\end{aligned}$$

Since f_T and w must stay with a distance $h/2$ from one another, the intermediate value theorem implies that there exists $u_{N(T)-2} \in [\bar{t}_{N(T)-2}, \bar{t}_{N(T)-1}]$ such that $f_T(u_{N(T)-2}) = \eta_T(u_{N(T)-2})$. Thus,

$$\begin{aligned}
\int_{\bar{t}_{N(T)-1}}^T |\eta'_T(u)|^\alpha du &\leq \int_{u_{N(T)-2}}^T |\eta'_T(u)|^\alpha du \\
&\leq \int_{u_{N(T)-2}}^T |f'_T(u)|^\alpha du \leq \int_{\bar{t}_{N(T)-2}}^T |f'_T(u)|^\alpha du
\end{aligned}$$

where we also used the fact that η_T minimizes the energy on $[0, T]$, and thus that η_T is also the only minimizer of

$$\begin{aligned}
&\text{Min} \left\{ \int_{u_{N(T)-2}}^T |\varphi'(u)|^\alpha du : \varphi \in \text{AC}([u_{N(T)-2}, T]), \right. \\
&\left. \varphi(u_{N(T)-2}) = f_T(u_{N(T)-2}), \varphi(T) = f_T(T), \|\varphi - w\|_{\infty, [u_{N(T)-2}, T]} \leq \frac{h}{2} \right\}.
\end{aligned}$$

Let us now assume that there exists K large enough such that $0 \leq \sigma_{\theta(T)-K} \leq \bar{t}_{N(T)-2}$. The previous inequality implies that

$$\begin{aligned} \int_{\bar{t}_{N(T)-1}}^T |\eta'_T(u)|^\alpha du &\leq \int_{\sigma_{\theta(T)-K}}^T |f'_T(u)|^\alpha du \\ &= \sum_{k=0}^{K-1} \int_{\sigma_{\theta(T)-(k+1)}}^{\sigma_{\theta(T)-k}} |f'_T(u)|^\alpha du + \int_{\sigma_{\theta(T)}}^T |f'_T(u)|^\alpha du \\ &\leq \frac{h^\alpha}{4^\alpha} \left(\sum_{k=0}^{K-1} \frac{1}{(\Delta\sigma_{\theta(T)-(k+1)})^{\alpha-1}} + \frac{1}{(T - \sigma_{\theta(T)})^{\alpha-1}} \right) \end{aligned}$$

where $\Delta\sigma_i = \sigma_{i+1} - \sigma_i$.

Step 3. By applying the same renewal theorem used in Step 0 (applied now to the sequence $\{\sigma_n\}_{n \geq 0}$), the RHS of the latter inequality is tight. Thus, we need to show that

$$\lim_{K \uparrow \infty} \lim_{T \uparrow \infty} P(0 \leq \sigma_{\theta(T)-K} \leq \bar{t}_{N(T)-2}) = 1.$$

Recall that $\Delta_n = \frac{4}{h}(w(\sigma_{n+1}) - w(\sigma_n))$. Let us assume that we can find $1 < i_1 < i_2 < i_3 < i_4 < K-3$ such that

$$\Delta_{\theta(T)-i_k} = \cdots = \Delta_{\theta(T)-(i_k-3)} = (-1)^k$$

so that w experiences a downcrossing (resp., upcrossing) of size h for k odd (resp. even) on the interval $[\theta(T) - i_k, \theta(T) - (i_k - 4)]$. By reasoning as in Lemma 4.2, the times $\sigma_{\theta(T)-i_k}$'s must all belong to distinct intervals $[\bar{t}_i, \bar{t}_{i+1}]$. Since $\theta(T) \leq T$, this yields

$$\sigma_{\theta(T)-i_4} \leq \bar{t}_{N(T)-2}.$$

On the other hand, by independence of the σ_k 's and the Δ_k 's, the sequence

$$(\Delta_{(\theta(T)-1)+}, \cdots, \Delta_{(\theta(T)-K)+}, \cdots)$$

converges in distribution to the first K coordinates of an infinite sequence of i.i.d. random variables $(\delta_1, \cdots, \delta_K, \cdots)$ with $P(\delta_i = \pm 1) = 1/2$. Since the probability to find 4 indices $1 < i_1 < \cdots < i_4$ such that

$$\delta_{i_k} = \cdots = \delta_{i_k-3} = (-1)^k$$

in the infinite sequence $\{\delta_k\}$ is equal to 1, it follows that

$$\lim_{K \rightarrow \infty} \lim_{T \rightarrow \infty} P(\sigma_{\theta(T)-K} \leq \bar{t}_{N(T)-2}) = 1.$$

□

Corollary 4.5. *The sequence $\{R(T)\}_{T \geq 0}$ is tight.*

Proof. From the devious result, it remains to control the first term of $R(T)$, i.e., $\int_0^{\bar{t}_2} c(\eta'_T(u))du$. When $N(T) \geq 4$, Theorem 1.1 implies that $\eta_T(\bar{t}_2) = w(\bar{t}_2) - h/2$ and thus the taut string η_T restricted on the interval $[0, \bar{t}_2]$ must be the unique solution of

$$\text{Min} \left\{ \int_0^{\bar{t}_2} |\varphi'(u)|^2 du : \varphi \in \text{AC}([0, \bar{t}_2]), \varphi(0) = 0, \varphi(\bar{t}_2) = w(\bar{t}_2) - \frac{h}{2}, \|\varphi - w\|_{\infty, [0, \bar{t}_2]} \leq \frac{h}{2} \right\}.$$

This implies that $\int_0^{\bar{t}_2} c(\eta'_T(u))du$ is constant, if T is large enough so that $N(T) \geq 4$. This obviously entails tightness of the first term. This ends the proof of our corollary. □

5. PROOF OF PROPOSITION 3.1

Let $[x]$ denote the integer part of x . Write

$$U_{\tilde{N}(t)-1} - t \frac{\bar{X}}{\bar{\tau}} = (U_{\tilde{N}(t)-1} - U_{[\frac{t}{\bar{\tau}}]}) + (U_{[\frac{t}{\bar{\tau}}]} - t \frac{\bar{X}}{\bar{\tau}}).$$

Our proposition is a direct consequence of the two following lemmas.

Lemma 5.1.

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \left((U_{\tilde{N}(t)-1} - U_{[\frac{t}{\bar{\tau}}]}) - \bar{X}(\tilde{N}(t) - [\frac{t}{\bar{\tau}}]) \right) = 0 \text{ in probability,}$$

Lemma 5.2. *The pair of random variables*

$$\left(\frac{1}{\sqrt{t}} (\tilde{N}(t) - t/\bar{\tau}) \cdot \frac{\bar{\tau}^{3/2}}{\sigma_{\tau}}, (U_{[\frac{t}{\bar{\tau}}]} - t \frac{\bar{X}}{\bar{\tau}}) \cdot \frac{\sqrt{\bar{\tau}/t}}{\sigma_X} \right)$$

converges in distribution to a two dimensional Gaussian random vector with mean 0 and covariance matrix $\Sigma = \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$, where $\rho = \sigma_{X,\tau}/\sigma_X\sigma_{\tau}$.

Proof of Lemma 5.1. Define $Z_n = U_n - \bar{X}n$. Let $\epsilon \in (0, 1)$ and define $n_0(t) = [t/\bar{\tau}]$, $n_1(t) = [t/\bar{\tau} \cdot (1 - \epsilon^3)]$, $n_2(t) = [t/\bar{\tau} \cdot (1 + \epsilon^3)]$. We aim at showing that $\frac{1}{\sqrt{t}} (Z_{n_0(t)} - Z_{\tilde{N}(t)-1})$ converges to 0 in probability.

$$\begin{aligned} P \left(\frac{1}{\sqrt{t}} |Z_{n_0(t)} - Z_{\tilde{N}(t)-1}| > \epsilon \right) &\leq P \left(\frac{1}{\sqrt{t}} |Z_{n_0(t)} - Z_{\tilde{N}(t)-1}| > \epsilon, \tilde{N}(t) \in [n_1(t), n_2(t)] \right) \\ &\quad + P \left(\tilde{N}(t) \notin [n_1(t), n_2(t)] \right) \end{aligned}$$

The second term on the RHS goes to 0 as n goes to ∞ by a standard renewal theorem. As for the first term, we use Kolmogorov inequality,

$$\begin{aligned} P \left(\frac{1}{\sqrt{t}} |Z_{n_0(t)} - Z_{\tilde{N}(t)-1}| \geq \epsilon, \tilde{N}(t) \in [n_1(t), n_2(t)] \right) &\leq P \left(\sup_{u \in \{n_1(t)-1, \dots, n_0(t)-1\}} \frac{1}{\sqrt{t}} |Z_u - Z_{n_0(t)}| \geq \epsilon \right) \\ &\quad + P \left(\sup_{u \in \{n_0(t)-1, \dots, n_2(t)-1\}} \frac{1}{\sqrt{t}} |Z_u - Z_{n_0(t)}| \geq \epsilon \right) \\ &\leq \frac{1}{\epsilon^2 t} \left(E(Z_{n_0(t)-n_1(t)}^2) + E(Z_{n_2(t)-n_0(t)}^2) \right) \\ &\leq \frac{2\sigma_X^2}{\epsilon^2 t} \frac{t}{\bar{\tau}} \epsilon^3 = 2 \frac{\sigma_X^2}{\bar{\tau}} \epsilon. \end{aligned}$$

Since the RHS goes to 0 with ϵ , this completes the proof of the lemma. \square

Proof of Lemma 5.2. To ease the notations, we write

$$Y_t = (U_{[\frac{t}{\bar{\tau}}]} - t \frac{\bar{X}}{\bar{\tau}}) \cdot \frac{\sqrt{\bar{\tau}/t}}{\sigma_X}$$

For any $y \in \mathbb{R}$ and $t \in \mathbb{R}^+$, define n_t^y to be the integer part of the unique positive solution (in x) of the equation

$$(7) \quad t = x\bar{\tau} + \sqrt{xy}\sigma_\tau.$$

Let $a < b, c < d$ be four arbitrary real numbers. First, for every y

$$n_t^y = \frac{t}{\bar{\tau}} - \sqrt{ty}\sigma_\tau/\bar{\tau}^{3/2} + o(\sqrt{t}).$$

and thus there exists $\epsilon_t^a, \epsilon_t^b$ such that $\epsilon_t^a/\sqrt{t}, \epsilon_t^b/\sqrt{t} \rightarrow \infty$ as $t \rightarrow 0$ and

$$(8) \quad \begin{aligned} P\left(\tilde{N}(t) \in [n_t^b, n_t^a], Y_t \in [c, d]\right) &= P\left(\tilde{N}(t) \in [t/\bar{\tau} - \frac{\sqrt{tb}\sigma_\tau}{\bar{\tau}^{3/2}} + \epsilon_t^b, t/\bar{\tau} - \frac{\sqrt{ta}\sigma_\tau}{\bar{\tau}^{3/2}} + \epsilon_t^a], Y_t \in [c, d]\right) \\ &= P\left(-\frac{1}{\sqrt{t}}(\tilde{N}(t) - t/\bar{\tau}) \cdot \frac{\bar{\tau}^{3/2}}{\sigma_\tau} \in [a - \frac{\tilde{\epsilon}_t^a}{\sqrt{t}}, b - \frac{\tilde{\epsilon}_t^b}{\sqrt{t}}], Y_t \in [c, d]\right) \end{aligned}$$

where $\tilde{\epsilon}_t^a/\sqrt{t}, \tilde{\epsilon}_t^b/\sqrt{t} \rightarrow 0$ as $t \rightarrow \infty$.

Secondly, let us now evaluate the law of the LHS of (8).

$$\begin{aligned} P\left(\tilde{N}(t) \in [n_t^b, n_t^a], Y_t \in [c, d]\right) &= P\left(S_{n_t^b} \leq t, S_{n_t^a} > t, Y_t \in [c, d]\right) \\ &= P\left(\frac{S_{n_t^b} - n_t^b\bar{\tau}}{\sigma_\tau\sqrt{n_t^b}} \leq \frac{t - n_t^b\bar{\tau}}{\sigma_\tau\sqrt{n_t^b}}, \frac{S_{n_t^a} - n_t^a\bar{\tau}}{\sigma_\tau\sqrt{n_t^a}} > \frac{t - n_t^a\bar{\tau}}{\sigma_\tau\sqrt{n_t^a}}, Y_t \in [c, d]\right) \\ &= P\left(\frac{S_{n_t^b} - n_t^b\bar{\tau}}{\sigma_\tau\sqrt{n_t^b}} \leq b, \frac{S_{n_t^a} - n_t^a\bar{\tau}}{\sigma_\tau\sqrt{n_t^a}} > a, Y_t \in [c, d]\right) \end{aligned}$$

where the third inequality follows directly from the definition of n_t^y . Since $n_t^a, n_t^b \approx t/\bar{\tau}$, it is tempting to use this approximation in the latter equality and write

$$P\left(\tilde{N}(t) \in [n_t^b, n_t^a], Y_t \in [c, d]\right) \approx P\left(\frac{S_{[t/\bar{\tau}]} - t}{\sigma_\tau\sqrt{t/\bar{\tau}}} \in (a, b], Y_t \in [c, d]\right),$$

More formally,

$$P\left(\tilde{N}(t) \in [n_t^b, n_t^a], Y_t \in [c, d]\right) = P\left(\frac{S_{[t/\bar{\tau}]} - t}{\sigma_\tau\sqrt{t/\bar{\tau}}} \leq b - \tilde{\epsilon}_t^b, \frac{S_{[t/\bar{\tau}]} - t}{\sigma_\tau\sqrt{t/\bar{\tau}}} > a - \tilde{\epsilon}_t^a, Y_t \in [c, d]\right)$$

where

$$\forall y = a, b, \quad \tilde{\epsilon}_t^y = \frac{1}{\sqrt{t}}(S_{n_t^y} - S_{[t/\bar{\tau}]} - \bar{\tau}(n_t^y - t/\bar{\tau})) + \frac{1}{\sqrt{t}}(S_{n_t^y} - \bar{\tau}n_t^y)\left(\sqrt{\frac{t/\bar{\tau}}{n_t^y}} - 1\right).$$

We now claim that $\tilde{\epsilon}_t^a$ and $\tilde{\epsilon}_t^b$ vanish as t goes to ∞ . Indeed, applying Markov inequality twice yields that for every $\delta > 0$

$$P(\tilde{\epsilon}_t^y > \delta) \leq \frac{1}{\delta^2 t} |n_t^y - [t/\bar{\tau}]| E(\tau^2) + \frac{n_t^y}{\delta^2 t} \left(\sqrt{\frac{t/\bar{\tau}}{n_t^y}} - 1\right)^2 E(\tau^2)$$

and thus $\lim_{t \uparrow \infty} \tilde{\epsilon}_t^y = 0$ in probability. On the other hand, by the multidimensional CLT, $\left(\frac{S_{[t/\bar{\tau}]} - t}{\sigma_\tau\sqrt{t/\bar{\tau}}}, Y_t\right)$ converges in distribution to the two dimensional gaussian vector with mean 0 and

correlation matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. It then follows that for every a, b, c, d

$$P\left(\tilde{\mathcal{N}}(t) \in [n_t^b, n_t^a], \left(U_{[\frac{t}{\tau}]} - t \frac{\bar{X}}{\tau}\right) \cdot \frac{\sqrt{\tau}/t}{\sigma_X} \in [c, d]\right) \rightarrow \int_{[a,b] \times [c,d]} \exp\left(-\frac{1}{2} {}^t x \Sigma^{-1} x\right) \frac{1}{2\pi \det(\Sigma)^{1/2}} d\lambda(x)$$

where λ is the Lebesgue measure on \mathbb{R}^2 . Combining this with (8) then yields our result. \square

6. APPENDIX

Proposition 6.1. *Let $0 < a < b$ and $c, d \in \mathbb{R}$ such that $|c - w(a)|, |d - w(b)| \leq h/2$. If C is a strictly convex function then both sets*

$$\begin{aligned} & \text{Argmin}\left\{\int_a^b C(\varphi'(t)) dt : \varphi \in AC([a, b]), \|w - \varphi\|_{\infty, [a, b]} \leq \frac{h}{2}\right\} \\ & \text{Argmin}\left\{\int_a^b C(\varphi'(t)) dt : \varphi \in AC([a, b]), \varphi(a) = c, \varphi(b) = d, \|w - \varphi\|_{\infty, [a, b]} \leq \frac{h}{2}\right\} \end{aligned}$$

have a unique element.

Proof. This is a rather standard result in convex analysis. For a proof, we refer to Lemma 2 in [G07]. In this reference, the result is shown in the particular case where $C(x) = \sqrt{1 + x^2}$ with fixed boundary conditions. We let the reader convince herself that the same proof applies for any convex function and also for the analogous minimization problem with free boundary conditions. \square

Proposition 6.2. *Let $0 < a < b$ and $c, d \in \mathbb{R}$ such that $|w(a) - c| \leq h/2$ and $|w(b) - d| \leq h/2$. Finally, let V be the unique solution of the minimization problem,*

$$\text{Min}\left\{\int_a^b |\varphi'(u)|^2 du : \varphi \in AC([a, b]), \varphi(a) = c, \varphi(b) = d, \|w - \varphi\|_{\infty, [a, b]} \leq \frac{h}{2}\right\}.$$

For any strictly convex function C , the function V is also the unique minimizer of the following minimization problem:

$$(9) \quad \text{Min}\left\{\int_a^b C(\varphi'(u)) du : \varphi \in AC([a, b]), \varphi(a) = c, \varphi(b) = d, \|w - \varphi\|_{\infty, [a, b]} \leq \frac{h}{2}\right\},$$

Before proceeding with the proof of the proposition, we note that an analogous result in the discrete setting can be found in [SGGHL09] – see Theorem 4.35 and Theorem 4.46 therein. As we shall now see, the latter proposition is also implicit in Grasmair [G07]. In the following, we fix a strictly convex function C and we will denote by U the unique solution of (9). We will now show that U and V must coincide.

Lemma 6.3. *If $u \in L^1([a, b])$ denotes the derivative of U , then u is of local bounded variation on (a, b) , i.e., for every $(s, t) \subset [a, b]$ the total variation of u on (s, t) is finite.*

Proof. We follow closely the proof of Proposition 2 in [G07]. We provide an argument by contradiction. Let us assume that there exists $t \in (a, b)$ such that for every $\delta > 0$, the total variation of u on $(t - \delta, t + \delta)$ is infinite. We distinguish between two cases: either Let us first assume that $U(t) < w(t) + \frac{h}{2}$ or $U(t) > w(t) - \frac{h}{2}$. We will only deal with the first case, since the second case is completely analogous.

We can find $s \in (t - \delta, t + \delta)$ such that

$$\text{ess inf}\{u(x), x \in (t - \delta, s)\} \leq \text{ess sup}\{u(x), x \in (s, t + \delta)\}$$

else u would be monotone and thus of finite variation. Thus, we can find $T_1 \subset (t - \delta, s)$ and $T_2 \subset (s, t + \delta)$ such that $|T_1| = |T_2| > 0$ (where $|T_i|$ denotes the Lebesgue measure of the set T_i) and

$$\text{ess inf}\{u(x), x \in T_1\} \leq \text{ess sup}\{u(x), x \in T_2\}.$$

Define

$$h(x) = \begin{cases} 1 & \text{if } x \in T_1 \\ -1 & \text{if } x \in T_2 \\ 0 & \text{otherwise} \end{cases}$$

and let $H(t) = \int_a^t h(s)ds$. Since $U(s) < w(s) + h/2$ on $(t - \delta, t + \gamma)$, for $\gamma > 0$ small enough, the function γH belongs to the set

$$\{Y \in AC(a, b) : Y(a) = 0, Y(b) = 0, \|U + Y - w\|_{\infty, [a, b]} \leq \frac{h}{2}\},$$

i.e., $U + \gamma H$ is an admissible function for the variational problem at hands. Further, the Gâteaux derivative of $\varphi \rightarrow \int_a^b C(\varphi'(x))dx$ evaluated at U in the direction H is given by

$$\int_{T_1} C'(u(x))dx - \int_{T_2} C'(u(x))dx.$$

(note the latter expression is well defined since C is a real convex function and thus is absolutely continuous). Since C' is strictly increasing, the choice of T_1 and T_2 induces that this derivative is strictly negative, which contradicts the minimality of U . This shows that for every t with $U(t) < w(t) + \frac{h}{2}$, there exists $\delta > 0$ such that the total variation of u on $(t - \delta, t + \delta)$ is finite. By a symmetric argument, one can show that the same property holds when t is such that $U(t) > w(t) + \frac{h}{2}$. This ends the proof of our lemma. \square

Since u is of local bounded variation, there exists a Radon measure Du satisfying the relation

$$\int_a^b \psi'(x)u(x)dx = - \int_a^b \psi(x)Du(dx)$$

for every function $\psi \in C_c^\infty(a, b)$ – the set of infinitely differentiable functions with compact support on (a, b) . Let $|Du|$ denote the total variation of Du (see again [G07] for more details). Since u is of local bounded variation (by the previous lemma), the Radon-Nikodym derivative $dDu/d|Du| \in L^1(a, b)$ is defined Du -almost surely on (a, b) , and takes the value ± 1 . (again Du almost surely). (For more details, we again we refer to [G07].)

Lemma 6.4. *U satisfies the following constraints*

- (1) $U \in AC(a, b)$.
- (2) $U(a) = c, U(b) = d$.
- (3) $\|U - w\|_{\infty, [a, b]} \leq h/2$.
- (4) u is of bounded local variation on (a, b) and further $U(t) = w(t) + \frac{h}{2} \frac{dDu}{d|Du|}(t)$ Du -a.s on (a, b) .

When u is a smooth function, $dDu/d|Du|$ coincides with $\text{sign}(u')$. Thus, the last condition can be interpreted as follows: away from the boundary of the tube, the function U is taut (i.e. $u' = 0$), whereas the only possibility for the function U to bend upwards (resp., downwards) is when U touches the upper part of the tube (resp., lower part of the tube), i.e., $U(t) = w(t) + \frac{h}{2}$ (resp., $U(t) = w(t) - \frac{h}{2}$).

Proof of Lemma 6.4. The first three properties directly follow from the definition of U . The previous lemma implies that u is of local bounded variation and it only remains to show that $U(t) = w(t) + \frac{h}{2} \frac{dDu}{d|Du|}(t)$ Du -a.s on (a, b) .

Again we follow closely Proposition 2 in [G07]. Let $t \in (a, b)$ be a Lebesgue point of the function $dDu/d|Du|$ (with respect to the measure $|Du|$) such that $dDu/d|Du| = 1$. We need to show that $U(t) = w(t) + \frac{h}{2}$. Let us assume that $U(t) < w(t) + \frac{h}{2}$. Since t is a Lebesgue point with respect to $|Du|$, we have

$$\lim_{\delta \rightarrow 0} \frac{\int_{t-\delta}^{t+\delta} |dDu/d|Du|(s) - 1| |Du|(ds)}{|Du|(t - \delta, t + \delta)} = 0$$

As a consequence, for δ small enough, we have

$$u(t + \delta) = u(t - \delta) + \int_{t-\delta}^{t+\delta} \frac{dDu}{d|Du|}(s) d|Du|(s) > u(t - \delta).$$

As in the previous lemma, this entails the existence of $T_1 \subset (t - \delta, t)$, $T_2 \subset (t, t + \delta)$ such that such that $|T_1| = |T_2| > 0$ and

$$\text{ess inf}\{u(x), x \in T_1\} \leq \text{ess sup}\{u(x), x \in T_2\}.$$

By the same reasoning as in the previous lemma, this contradicts the minimality of U , and thus $U(t) = w(t) + \frac{h}{2}$.

By the same argument, one can show if $t \in (a, b)$ a Lebesgue point of the function $dDu/d|Du|$ with respect to the measure $|Du|$ such that $dDu/d|Du| = -1$, then $U(t) = w(t) - \frac{h}{2}$. This completes the proof of our lemma. \square

Proof of Proposition 6.2. Using Proposition 3 in [G07], there is a unique function satisfying the constraints (1)–(4) of Lemma 6.4. On the other hand, those constraints do not depend on the strictly convex function C at hands. Since $x \rightarrow x^2$ is also strictly convex, it follows that $U = V$. \square

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